Section 9.3
The Quadratic Formula

The quadratic formula is a way of solving all quadratic equations to find their *roots*. The quadratic formula is derived by the general use of *completing the square*.

If we take the general form of the quadratic equation $\boldsymbol{ax^2 + bx + c = 0}$

and use *completing the square* to solve for *x*, then the quadratic formula is born.

STEP ONE: Divide whole equation by *a* to reduce the leading coefficient to 1.

$$\frac{a}{a}x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad \longrightarrow \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

STEP TWO: Move $\frac{c}{a}$ to the right side of the equation, leaving an empty space where the $\frac{c}{a}$ was:

$$x^2 + \frac{b}{a}x \qquad = -\frac{c}{a}$$

STEP THREE: Take one-half of the middle term $\left(\frac{b}{a}\right)$ and square it. *Complete the trinomial square* with this expression by placing it into the empty space (one-half of $\frac{b}{a}$ is $\frac{b}{2a}$, and $\frac{b}{2a}$ squared is $\frac{b^2}{4a^2}$).

Because $\frac{b^2}{4a^2}$ was added to the left-hand side, then $\frac{b^2}{4a^2}$ must be added to the right hand side:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

STEP FOUR: Factor the left hand-side of the equation as a *perfect trinomial square* and add the right-hand side:

FACTOR LEFT SIDE

$$\sqrt{x^2} = x \qquad \sqrt{\frac{b^2}{4a^2}} = \frac{b}{2a}$$

ADD RIGHT SIDE

$$-\frac{c}{a} + \frac{b^2}{4a^2} = \frac{-4ca + b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2} \quad \longrightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

STEP FIVE: Find the square root of both sides

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

STEP SIX: solve for x by moving $\frac{b}{2a}$ to the right hand side: $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$

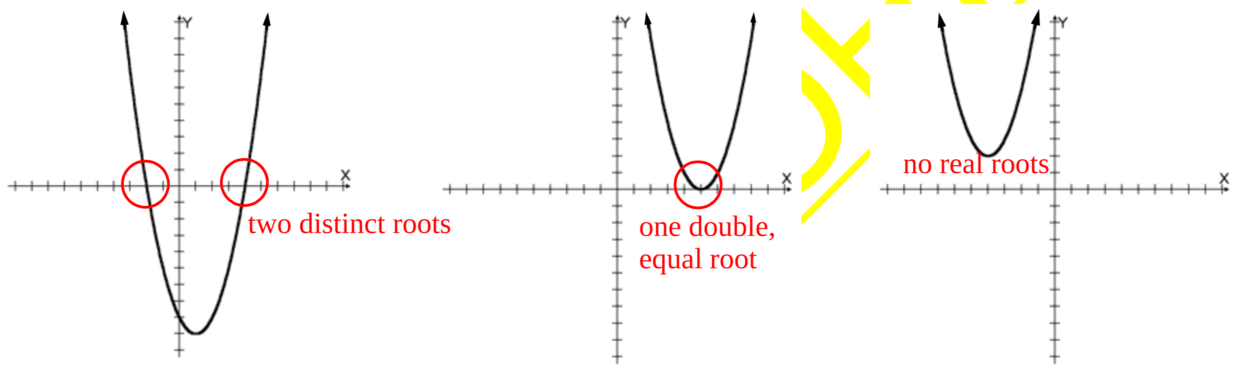
This is the formula for quadratic roots:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where ***a*** is the leading coefficient (quadratic term), ***b*** is the coefficient of the middle (linear) term, and ***c*** is the last (constant) number.

UNDERSTANDING WHAT THE ROOTS MEAN

In section 8.1 the shape of the quadratic line was given as the parabolic curve crossing the *x*-axis. Below left, the curve crosses the *x*-axis twice (two distinct roots), below center the curve does not cross but touches the *x*-axis (one double, equal root), below right the curve does not cross the *x*-axis (no real roots). Depending on the ***a***, ***b*** and ***c*** values of the quadratic formula, the shape and position of the curve is determined.



Example (for two distinct roots): Solve $2x^2 + 5x - 1 = 0$ $\boldsymbol{a} = 2$ $\boldsymbol{b} = 5$ $\boldsymbol{c} = -1$

Substitute in the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-5 \pm \sqrt{(5)^2 - 4(2)(-1)}}{2(2)} \rightarrow x = \frac{-5 \pm \sqrt{25 + 8}}{2(2)} \rightarrow x = \frac{-5 \pm \sqrt{33}}{4}$$

$$x = \frac{-5 + 5.745}{4} \rightarrow x = \frac{-5 - 5.745}{4}$$

$x =$ **0.186** $x =$ **–2.686**

Example (for one double, equal root): Solve $x^2 - 5x + 6.25 = 0$ $\boldsymbol{a} = 1$ $\boldsymbol{b} = -5$ $\boldsymbol{c} = 6.25$

Substitute in the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-(-5)\pm\sqrt{(-5)^2 - 4(1)(6.25)}}{2(1)} = x = \frac{5 \pm \sqrt{25-25}}{2(1)} = x = \frac{5 \pm \sqrt{0}}{2} = x = \frac{5}{2} = x = \mathbf{2.5}$$

Example (for no roots): Solve $x^2 + 2x + 6 = 0$ $\quad$ $\boldsymbol{a} = 1$ $\quad$ $\boldsymbol{b} = 2$ $\quad$ $\boldsymbol{c} = 6$

Substitute in the quadratic formula: $\quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(6)}}{2(1)} \rightarrow x = \frac{-2 \pm \sqrt{4-24}}{2} \rightarrow x = \frac{-2 \pm \sqrt{-20}}{2}$$

Because the square root of a negative number cannot be determined, the answer is not real.

The Discriminant: $\sqrt{b^2 - 4ac}$

The discriminant is that part of the quadratic formula that allows us to peek into it to see if the quadratic has any valid roots. If the discriminant is negative, no need to continue; if it is positive or zero, continue.

Example: Solve $7x^2 - 13x + 13 = 0$ $\quad$ $\boldsymbol{a} = 7$ $\quad$ $\boldsymbol{b} = -13$ $\quad$ $\boldsymbol{c} = 13$

Using the discriminant $\sqrt{(-13)^2 - 4(7)(13)} = \sqrt{169-364} = \sqrt{-195}$ $\quad$ Radical is negative, zero roots.

Practice:
Using the quadratic formula, find the roots of the quadratic.

1. $x^2 + 8x - 4 = 0$
2. $x^2 - 10x - 8 = 0$
3. $2x^2 + 13x - 5 = 0$
4. $x^2 + 5x - 1 = 0$
5. $3x^2 + 9x - 5 = 0$
6. $5x^2 - 7x - 18 = 0$
7. $x^2 + 11x - 13 = 0$
8. $7x^2 + 3x - 44 = 0$
9. $x^2 - 12x + 1 = 0$
10. $5x^2 + x - 8 = 0$
11. $x^2 - x - 20 = 0$
12. $12x^2 + 12x - 1 = 0$
13. $3x^2 - 7x - 22 = 0$
14. $2x^2 + 14x + 3 = 0$
15. $2x^2 - 5x - 12 = 0$
16. $3x^2 - 13x - 28 = 0$
17. $5x^2 + 17x + 3 = 0$
18. $x^2 + 22x - 81 = 0$
19. $4x^2 - 17x + 2 = 0$
20. $3x^2 - 3x - 23 = 0$
21. $2x^2 - x - 15 = 0$

Use the discriminant to determine if the quadratic has roots.

22. $x^2 + 2x + 8 = 0$
23. $2x^2 + 2x + 13 = 0$
24. $7x^2 + 2x - 1 = 0$
25. $-2x^2 + 2x + 11 = 0$
26. $3x^2 + 5x + 21 = 0$
27. $-3x^2 + 22x - 8 = 0$
28. $12x^2 - 13x + 8 = 0$
29. $3x^2 + 15x - 2 = 0$
30. $x^2 + 6x + 15 = 0$
31. $5x^2 + 14x - 2 = 0$
32. $5x^2 + x + 9 = 0$
33. $3x^2 + 5x - 11 = 0$
34. $2x^2 + 7x + 14 = 0$
35. $22x^2 + 100x + 108 = 0$
36. $x^2 + 9x + 50 = 0$