Section 9.2
Completing the Square

So far we have solved quadratic equations that can be factored, like:

$$x^2 - 5x - 6 = 0$$

Determine: Which two integers give a product of –6 and a difference of –5?

$$(x - 6)(x + 1) = 0$$
$$x - 6 = 0 \qquad\qquad x + 1 = 0$$
$$x = 6 \text{ and } -1$$

But what happens when we encounter a quadratic equation which cannot be factored?

Example: $x^2 - 4x - 7 = 0$

Determine: Which two integers give a product of –7 and a difference of –4?

The factors of 7 are 1 and 7

$$x^2 - 4x - 7 = 0$$

The differences of 7 and 1 are 6 or –6, not –4

Because 7 has only two factors (7 and 1), and these two factors subtracted are 6 or –6 and not –4, like the middle term above, the answers to this quadratic are not integers, but decimals. Finding which combination is correct would be a very lengthy and difficult task.

To solve this quadratic we turn to a process called, **completing the square**, which involves making a *perfect trinomial square* out of the left side of the equation.

$$x^2 - 4x - 7 = 0$$

STEP ONE: Move –7 to the right side of the equation, leaving an empty space where the –7 was:

$$x^2 - 4x \quad = 7$$

STEP TWO: Take one-half of the middle term (–4) and square it. *Complete the trinomial square* with this number by placing it into the empty space (one-half of –4 is –2, –2 squared is 4).

Because 4 was added to the left-hand side, then 4 must be added to the right-hand side:

$$x^2 - 4x + 4 = 7 + 4$$

STEP THREE: Factor the left-hand side of the equation as a *perfect trinomial square*:

$$(x-2)(x-2) = 11$$
$$(x-2)^2 = 11$$

STEP FOUR: Find the square root of both sides: $\sqrt{(x-2)^2} = \sqrt{11} \rightarrow x - 2 = \pm 3.317$

Answer: $x = 2 + 3.317 =$ **5.317**, $x = 2 - 3.317 =$ **–1.317**

HOW TO SOLVE A QUADRATIC WITH A LEADING COEFFICIENT GREATER THAN ONE

Example: Solve $2x^2 - 8x - 9 = 0$

Because to *complete a square* the leading coefficient MUST be one, divide the whole equation by 2:

$$\frac{2x^2}{2} - \frac{8x}{2} - \frac{9}{2} = 0 \rightarrow x^2 - 4x - 4.5 = 0$$

Move –4.5 to the right: $x^2 - 4x = 4.5$

Complete the square: $x^2 - 4x + 4 = 4.5 + 4$
$x^2 - 4x + 4 = 8.5$

Factor: $(x-2)(x-2) = 8.5$

Find square root: $\sqrt{(x-2)^2} = \sqrt{8.5}$

$x - 2 = \pm 2.915$

$x = 2 + 2.915 =$ **4.915** $x = 2 - 2.915 =$ **–0.915**

Example: Solve $3x^2 + 5x - 11 = 0$

Divide equation by 3 $\frac{3x^2}{3} + \frac{5x}{3} - \frac{11}{3} = 0$

Move $-\frac{11}{3}$ to the right $x^2 + \frac{5}{3}x = \frac{11}{3}$

Complete the square (one-half of $\frac{5}{3}$ is $\frac{5}{6}$ and $\frac{5}{6}$ squared is $\frac{25}{36}$) and add to both sides of the equation:

$$x^2 + \frac{5}{3}x + \frac{25}{36} = \frac{11}{3} + \frac{25}{36}$$

Add right-hand side $$x^2 + \frac{5}{3}x + \frac{25}{36} = \frac{157}{36}$$

Factor left-hand side as a *perfect trinomial square* and find square root of both sides:

$$\sqrt{\left(x + \frac{5}{6}\right)\left(x + \frac{5}{6}\right)} = \sqrt{\frac{157}{36}}$$

$$x + \frac{5}{6} = \pm 2.088$$

Solve for x: $x = -0.8\overline{3} + 2.088 = \mathbf{1.255}$ $x = -0.8\overline{3} - 2.088 = \mathbf{-2.921}$

Practice:
Find the roots of the quadratic by completing the square.

1. $x^2 + 2x - 8 = 0$
2. $y^2 + 10y - 24 = 0$
3. $x^2 + 6x - 12 = 0$
4. $x^2 + 8x - 20 = 0$
5. $x^2 - 12 = 0$
6. $x^2 + 3x = 54$
7. $y^2 + 5y - 36 = 0$
8. $a^2 + 7a - 8 = 0$
9. $2x^2 + 4x - 6 = 0$
10. $2y^2 + 8y = 10$
11. $3a^2 + 12a - 15 = 0$
12. $3r^2 + 24r - 27 = 0$
13. $2x^2 + 7x - 15 = 0$
14. $2y^2 + 6y = 8$
15. $2x^2 + 9x - 5 = 0$
16. $2a^2 + 10a + 8 = 0$
17. $x^2 + 4x + 1 = 0$
18. $y^2 + 8y - 7 = 0$
19. $x^2 + 12x + 5 = 0$
20. $x^2 + 20x - 9 = 0$
21. $y^2 + 16y - 12 = 0$
22. $x^2 + 12x - 11 = 0$
23. $a^2 + 24a - 8 = 0$
24. $y^2 + 8y + 4 = 0$
25. $x^2 + 24x - 13 = 0$
26. $x^2 + 32x - 12 = 0$
27. $y^2 + 8y + 2 = 0$
28. $a^2 + 20a - 4 = 0$
29. $x^2 + 4x - 15 = 0$
30. $y^2 + 16y - 6 = 0$
31. $x^2 + 24x + 8 = 0$
32. $x^2 + 4x - 10 = 0$
33. $2x^2 + 7x - 14 = 0$
34. $3y^2 + 13y + 6 = 0$
35. $5x^2 + 9x - 20 = 0$
36. $4p^2 + 3p - 16 = 0$
37. $2y^2 + 15y + 8 = 0$
38. $2m^2 + 10m - 12 = 0$
39. $3x^2 - 15x - 18 = 0$
40. $3y^2 - 9y - 16 = 0$
41. $2x^2 + 17x + 9 = 0$
42. $5w^2 - w - 4 = 0$
43. $2y^2 - 21y - 10 = 0$
44. $5a^2 + 3a - 10 = 0$
45. $2x^2 - 7x - 14 = 0$
46. $3y^2 - 14y - 21 = 0$
47. $2c^2 + 9c - 9 = 0$
48. $2d^2 - 25d + 15 = 0$