Section 7.6
Factoring by Grouping

Factoring by grouping is a way of breaking down an expression into more factorable expressions.

Example: Factor $x^3 + x^2 + x + 1$

The expression above does not have a common factor; it is not a *difference of squares* and it is not a *trinomial*. However, the expression can be separated into two binomials.

$$(x^3 + x^2) + (x + 1)$$

Factoring x^2 from the left binomial: $x^2(x + 1)$

The new expression becomes: $x^2(x + 1) + (x + 1)$ or $x^2(x + 1) + 1(x + 1)$

Notice now that the binomial $(x + 1)$ is a common factor within the brackets.

Factoring again, this time $(x + 1)$: $(x + 1)(x^2 + 1)$

Example: Factor $12y^3 - 4y^2 + 15y - 5$

Splitting in two binomials $(12y^3 - 4y^2) + (15y - 5)$

Factoring binomials separately $4y^2(3y - 1) + 5(3y - 1)$

Factoring the common factor $(3y - 1)$: $(3y - 1)(4y^2 + 5)$

Example: Factor $a^2c^2 + bc^2 - 2a^2 - 2b$

Splitting in two binomials $(a^2c^2 + bc^2) + (-2a^2 - 2b)$

Factoring binomials separately (notice that in the second binomial the negative sign was factored also).

$$c^2(a^2 + b) - 2(a^2 + b)$$

Factoring the common factor $(a^2 + b)$: $(a^2 + b)(c^2 - 2)$

Example: Factor $4x^2 - 12xy + 9y^2 - z^2$

In this particular example, splitting the polynomial in two binomials will serve no purpose; however, because there is a *perfect trinomial square* and a negative square at the end, the split will leave a trinomial and $(-z^2)$.

$$(4x^2 - 12xy + 9y^2) - z^2$$

Factoring the trinomial: $(2x - 3y)(2x - 3y) = (2x - 3y)^2$

The *perfect trinomial square* becomes the first term of a *difference of two squares*, where z^2 is the second term

$$(2x - 3y)^2 - z^2$$

Factoring the *difference of two squares:* $[(2x - 3y) + z][(2x - 3y) - z]$

Eliminating parentheses: $[2x - 3y + z][2x - 3y - z]$

Example: Factor $x^6 - x^4 - x^2 + 1$

Splitting in two binomials $(x^6 - x^4) + (-x^2 + 1)$

Factoring binomials separately (notice that in the second binomial the negative sign was factored also).

$$[x^4(x^2 - 1)] - [1(x^2 - 1)]$$

Factoring the common factor $(x^2 - 1)$: $(x^2 - 1)(x^4 - 1)$

Factor both binomials (*difference of two squares*): $(x + 1)(x - 1)(x^2 + 1)(x^2 - 1)$

Last term is still a *difference of two squares*. Continue to factor: $(x + 1)(x - 1)(x^2 + 1)(x + 1)(x - 1)$

Rearrange in descending order and square like binomials: $(x^2 + 1)(x + 1)^2(x - 1)^2$

Practice:
Factor completely.

1. $y^3 + 5y^2 + 2y + 10$
2. $6x^3 - 3x^2 + 10x - 5$
3. $x^3 + 2x^2 + 3x + 6$
4. $x^3 + 3x^2 + 4x + 12$
5. $3a^2c^2 + 6bc^2 - 4a^2 - 8b$
6. $x^2 + 4xy + 4y^2 - 9$
7. $5x^3 + 10x^2 + 3x + 6$
8. $3y^3 + 2y^2 - 12y - 8$
9. $14y^3 - 6y^2 + 21y - 9$
10. $9x^2 - 6xy + y^2 - 4$
11. $6a^2c^2 + 3bc^2 + 2a^2 + b$
12. $x^6 - x^4 - x^2 + 1$
13. $5y^3 - 40y^2 + 7y - 56$
14. $y^3 + 6y^2 - 5y - 30$
15. $12a^2c^2 + 8bc^2 - 15a^2 - 10b$
16. $16x^2 + 8xy + y^2 - 25$
17. $36x^6 - 9x^4 - 4x^2 + 1$
18. $6a^2c^2 + 8bc^2 - 9a^2 - 12b$
19. $8y^3 - 8y^2 - 9y + 9$
20. $x^2 - 10xy + 25y^2 - w^2$
21. $x^6 - 4x^4 - 9x^2 + 36$
22. $25a^2c^2 - 5bc^2 + 10a^2 - 2b$
23. $16x^6 - 4x^4 - 4x^2 + 1$
24. $14a^2c^2 + 35bc^2 - 6a^2 - 15b$
25. $6y^3 - 10y^2 + 21y - 35$
26. $3y^3 - 15x^2 + 2x + 10$
27. $64x^2 - 144xy - 4x + 9y$
28. $x^3 + 3x^2 + 5x + 15$
29. $12a^2c^2 + 3bc^2 + 4a^2 + b$
30. $10x^3 - 6x^2 + 35x - 21$
31. $y^3 + 4y^2 + 7y + 28$
32. $x^3 - 2x + 5x^2 - 10$
33. $20y^3 + 15y^2 + 24y + 18$
34. $a^2 + 12a + 36 - b^2$
35. $25y^2 - 30y + 9 - x^2$
36. $x^3 - 9x - 4x^2 + 36$