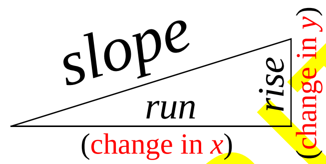
Section 3.3
The Slope

In section 3.2, linear equations and the use of the slope were introduced. Now the question will be how to come up with the slope and its corresponding linear equation without the use of graphs. In other words, analytically: using symbols and numbers only.

The diagram shows a ramp where the slope is defined as the constant ratio (if the slope is constant the line will always be straight) of the change in *y* over the change in *x*, or:



$$slope = m = \frac{change\ \ of\ \ y}{change\ \ of\ \ x}$$

If we try to find a *change* for *y*, we need to know where *y* begins and where it ends, and, of course, the same must be done for *x*: We need to know where *x* begins and where it ends.
To do this we use the coordinate values of two points on the line, then subtract the coordinates' values of *y* and the coordinates' values of *x*. The equation to find the slope is:

$$\text{slope} = m = \frac{y_2 - y_1}{x_2 - x_1}$$

x_1 y_1 x_2 y_2

Example: The line beginning or passing through point (2, 5) and ending or passing through point (–3, –5) has the slope

$$\frac{-5-5}{-3-2} = \frac{-10}{-5} = 2$$

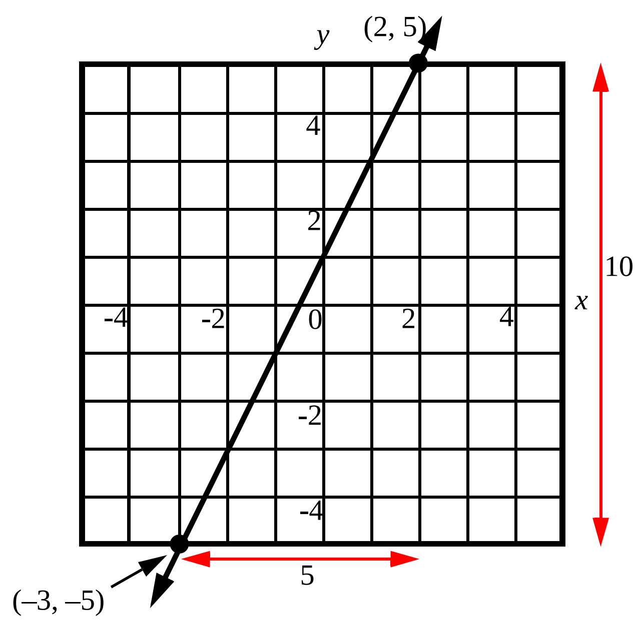
It doesn't matter which point is labeled 1 or which point labeled 2, as long as we don't mix the coordinates. Doing it backwards you get the same result:

$$\frac{5-(-5)}{2-(-3)} = \frac{10}{5} = 2$$

In the graph to the right, notice that the vertical (up and down) difference between the two points is 10: This is the difference of the values of ***y***. And the horizontal distance between the two points is 5: This is the difference of the values of ***x***. If the *y* distance is 10, and the *x* distance is 5:

$$slope = m = \frac{change\ \ of\ \ y}{change\ \ of\ \ x} = \frac{10}{5} = 2$$

It agrees with the above work that the slope is 2.



Example: Find the slope of the line passing through points (5, –4) and (7, 8).

If $y_2 = 8$ and $y_1 = -4$ then $8 - (-4)$
If $x_2 = 7$ and $x_1 = 5$ then $7 - 5$

Write it as the slope equation using ***m*** for the slope: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Then: $m = \dfrac{8 - (-4)}{7 - 5} = \dfrac{12}{2} = 6$

Example: Find the slope of the line passing through points (–5, 6) and (2, –8).

$$m = \frac{-8 - 6}{2 - (-5)} = \frac{-14}{7} = -2$$

FINDING THE EQUATION OF A LINE WHEN THE SLOPE AND A POINT ARE KNOWN

Because the slope is part of an equation, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, we can substitute the slope and one of the points in the equation to find the ***y-intercept*** equation (see section 3.2). In the example above ***m* = –2** and using the second point **(2, –8)**, the slope equation becomes:

$-2 = \dfrac{y - (-8)}{x - 2}$ Solving the equation for *y*: $\dfrac{-2}{1} = \dfrac{y + 8}{x - 2}$

Multiplying across: $y + 8 = -2(x - 2)$

Distributing the right side: $y + 8 = -2x + 4$

Subtracting 8 from both sides to isolate ***y***: $y + 8 - 8 = -2x + 4 - 8$

$y = -2x - 4$

Where the *slope*, *m*, is –2 and the *y-intercept* is –4.

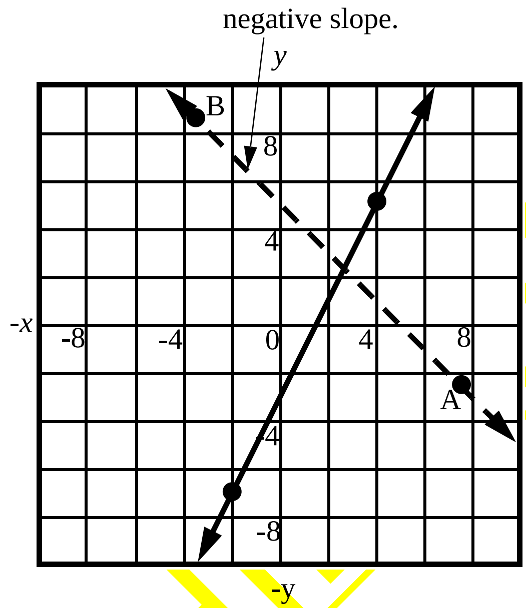
SLOPE, LINES, TABLES AND EQUATIONS

The idea that a linear relationship can be expressed as a line is based on the fact that every line has a constant slope (it has to go somewhere) and points (coordinates) on a line can be paired on a table. From there, it follows that a mathematical relationship (equation) may be set up to show that for every value of ***x***, there is one and only one corresponding value for ***y*** (because it is a straight line).

Example:

The points (4,5) and (–2,–7) form a line. Plot the points, write the *y*-intercept equation for the line, and find three other points on the line.

There are various way to approach the solution. If we are good at drawing, we can try for a graphical solution.

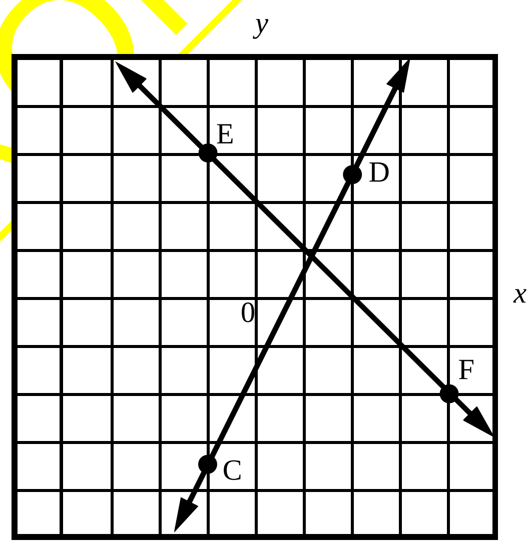


In the graphical solution the points are plotted and the line drawn. From the line, we select the value of *y* when *x* is zero (*y*-intercept). In this case the line crosses the *y*-axis at –3, so the *y*-intercept is –3. Next, find the slope. Because the slope is $\frac{y}{x}$, and the *y* distance between the two points is 12 and the *x* distance between the two points is 6, then the slope is $\frac{12}{6} = 2$. Now determine if the slope is positive or negative. We do this by looking at the line. **If the line slopes UP from LEFT TO RIGHT, the slope is positive, like in this case, but if the line slopes UP from RIGHT TO LEFT, the slope is negative**. The dashed line on the graph shows this.

Having determined that the *slope* is 2 and the *y*-intercept is –3, we are ready to write the equation for the line:

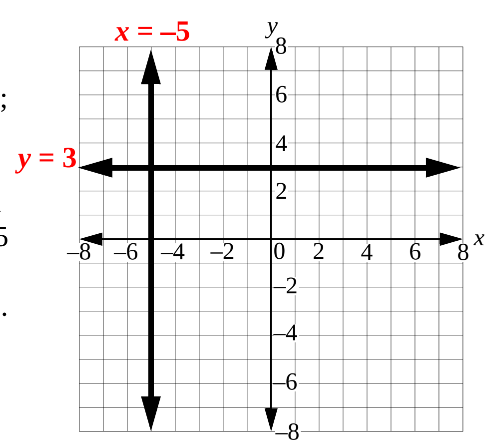
$$y = 2x - 3$$

slope   *y*-intercept



Another way to determine the slope: (See second graph.) Moving from point C to point D (up and to the right), the *x* and *y changes* are +, +, which means that the slope is positive.

Moving backwards from point D to point C, the *x* and *y changes* are –,–. Therefore, the slope shows again that it is positive. If you try the same approach with line $\overline{EF}$, the *x* and *y changes* from E to F will be +,– and from F to E –,+, thus, in both instances, the slope will be negative.



There are two situations where the value of the slope is easily identifiable: When the line is horizontal and when the line stands up completely vertical. When the line is completely flat there is no value for *y* (no **rise**) and, therefore, the slope is *zero*; when the line is vertical there is no value for *x* (no **run**) and, therefore, because the slope is so large, it cannot be defined. Because of this, equations for horizontal lines contain no *x* and the equations for vertical lines contain no *y*. For example, *x* = –5 means that this is a vertical line that comes down through –5. On the other hand, *y* = 3 is a horizontal line that goes through 3. In the last graph, you can see both examples.

To relate all of this to three different values—or points—for the same line, build a table of corresponding values. Notice that the given point (4,5) is there (in red); the other points were found by the use of the slope $\frac{6}{3} = \frac{2}{1}$, which says that the value of y will change 6 times for every time x changes 3 times, or by reducing the ratio to 2 over 1, the value of y will change 2 times for every time that x changes once. Therefore, notice how the value of x goes 2, 3, 4, 5 (changes by 1) and the value of y goes 1, 3, 5, 7 (changes by 2). Why? Because the slope is 2.

x	y
2	1
3	3
4	**5**
5	7

Example: Plot line $3x + 4y = 12$.

There are two ways to answer this problem: By building a table and graph, or by writing the y-intercept form.

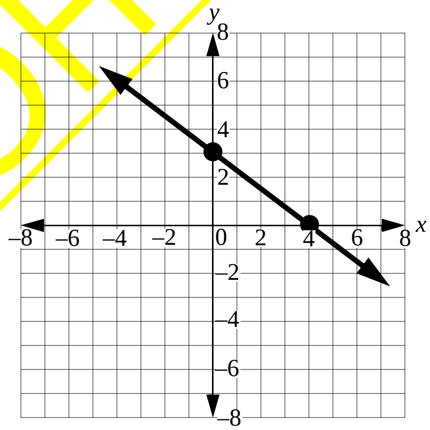
BUILDING A TABLE

Using a table we find two points. The most convenient ones are the y- and x-intercepts (when $x = 0$, and when $y = 0$). Although plotting more points increases line accuracy when you build a table, two points are sufficient to just draw a line, and the y- and x-intercepts happen to be the easiest to get.

When $x = 0$
$$3(0) + 4y = 12$$
$$4y = 12$$
$$y = 3$$

When $y = 0$
$$3x + 4(0) = 12$$
$$3x = 12$$
$$x = 4$$

x	y
0	**3**
1	2.25
2	1.5
3	0.75
4	**0**

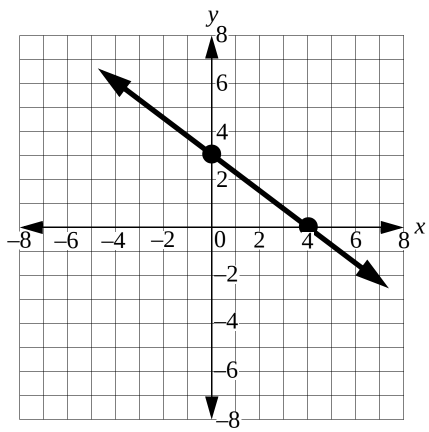


Using the y-intercept:

$$3x + 4y = 12$$
$$4y = -3x + 12 \qquad (-3x \text{ on both sides})$$
$$y = \frac{-3x}{4} + 3 \qquad (\text{divide by 4 both sides})$$

slope y-intercept

The *slope-intercept* equation shows that the y-intercept is 3 and the slope is $\frac{-3}{4}$. Plot the y-intercept (what's the value of y when x is zero) first by going to the origin (0,0) and moving up +3 along the y-axis. Then, using the slope, we plot the second point by starting from (0,3), go down 3 (the y value of the slope is negative, up if the slope were positive) and to the right (the x value of the slope) 4.



Notice that both graphs are identical, yet plotted differently.

Example: Is point (–7,3) a solution to line $4x + y = 18$?

To say that a point is a solution is the same as saying the point is ON the line. And to find out if the point is ON the line, the coordinates of the points, in this particular case $x = -7$ and $y = 3$, must make the equation true: *Left side = Right side*.

$$4(-7) + 1(3) = -25 \quad \text{(NOT a solution)}$$

x y

Trying another point, (6,–6)

$$4(6) + 1(-6) = 18 \quad \text{(It is a solution)}$$

Example: Write the equation in *standard* form for a line that crosses point (3,–5) and has slope $-\frac{1}{3}$.

Using the slope equation $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, if the slope is $-\frac{1}{3}$ and the point $(\overset{x}{3},\overset{y}{-5})$ then: $-\dfrac{1}{3} = \dfrac{y-(-5)}{x-3}$ $\begin{matrix}\text{point } y\\ \text{point } x\end{matrix}$

Multiplying across: $3(y + 5) = -1(x - 3)$

Distributing: $3y + 15 = -x + 3$

Subtracting 15 on both sides: $3y + 15 - 15 = -x + 3 - 15$

$3y = -x - 12$

Writing it in standard form: $x + 3y = -12$

"Standard" form means A*x* + B*y* = C, where both "*x*" and "*y*" are to the left of the equal sign. Compare this to the "slope-intercept" form, where "*y*" and "*x*" are to either side of the equal sign (*y* = *mx* + *b*), and "general" form, where all three values are equal to zero (A*x* + B*y* + C = 0).

Practice:

Find the slope and *y*-intercept.

1. $y = 3x + 5$
2. $y = -\frac{1}{4}x - 2$
3. $y = 2.5x - 9$
4. $y = -1.8x - 7$
5. $y = x - 8$
6. $y = -x + 5$
7. $y = \frac{2}{3}x - 2$
8. $y = -7x + \frac{1}{2}$
9. $x + y = 6$
10. $3x - 4y = 12$
11. $y - x = 3$
12. $-y - x + 5 = 0$
13. $-2x - 5y = 15$
14. $y + 3x = 8$
15. $2.4x - 1.2y + 3.6 = 0$
16. $0 = 5 - x + y$
17. $2x + 3y = 6$
18. $\frac{2}{3}x + \frac{1}{4}y = 12$
19. $\frac{1}{2}y + x + 8 = 0$
20. $5.2 + 1.3x - 2.6y = 0$
21. $2x - 7y = 14$

Write the slope-intercept equation ($y = mx + b$) that contains each set of points shown.

22. (–7,–5), (4,6)
23. (3,1), (–2,–2)
24. (1,–6), (4,8)
25. (–4,5), (6,–3)
26. (–2,1), (8,2)
27. (–8,–5), (1,–5)
28. (3,5), (3,–6)
29. (–5,–8), (–1,–1)
30. (9, 3), (–8,–1)

Write the standard equation ($Ax + By = C$) that contains the points and slope given.

31. (1,7) $m = 2$
32. (8,–4) $m = -3$
33. (–5,2) $m = -\frac{1}{4}$
34. (9,–6) $m = \frac{2}{3}$
35. (3,9) $m = -4$
36. (–4,3) $m = 6$
37. (0.5,–1.5) $m = 0$
38. (0,0)), $m = -1$
39. (–5,–2) $m = \frac{5}{4}$

Write a table for each equation. From the table, find the slope and y-intercept.

40. $x + y = 3$
41. $2x - 3y = 8$
42. $4y + 5x + 10 = 0$
43. $y + 3x = 6$
44. $\frac{1}{2}x + \frac{3}{4}y = 8$
45. $x - y = 7$
46. $5a - 2b + 20 = 0$
47. $5x + 3y = -1$
48. $7x - 5y = 20$

Find the slope and y-intercept. Plot the line and label it using either the standard or slope-intercept form.

49.

x	y
2	–2
3	0
4	2
5	4

50.

x	y
–8	7
–6	13
–4	19
–2	25

51.

x	y
0	3
–2	6
–4	9
–6	12

52.

x	y
0.1	1.5
0.8	2.9
1.5	4.3
2.2	5.7

53.

x	y
–1	–4
0	–1
1	2
2	5

54.

x	y
–3	2
–7	–3
–11	–8
–15	–13

55.

x	y
12	–9
9	–6
6	–3
3	0

56.

x	y
1.5	–4.5
3	–3
4.5	–1.5
6	0

57.

x	y
5	5
2	7
–1	9
–4	11

58.

x	y
0	0
–2	5
–4	10
–6	15

59.

x	y
4	1
3	2
2	3
1	4

60.

x	y
7	9
14	5
21	1
28	–3